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Irrotational vector fields, partially invariant with respect to group of rotations☆

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ABSTRACT

A geometrical description of three-dimensional irrotational vector fields which are partially invariant with respect to the $SO(3)$ group of rotations is given. It is shown that the motion of a continuum, that satisfies the equation of continuity and is partially invariant with respect to $SO(3)$, is necessarily rotational, that is, the curl of the velocity vector is non zero.

The submodel of the equations of gas dynamics, known as the "singular vortex" or the "Ovsyannikov vortex", describes the exact reduction of the equations of gas dynamics as a partially invariant solution constructed for the $SO(3)$ group of rotations. A solution of this type was derived for the first time by Ovsyannikov,[1] and also the existence of a solution was proved, a system of equations defining a submodel was found and the basic properties of a gas flow, specified by the solution, were established. The notion "singular vortex", which initially referred solely to a special class of solutions with initial data determined on an entire sphere, became used for every class of solutions which are partially invariant with respect to a group of rotations.

In the case of the equations of an ideal incompressible fluid, a solution which is partially invariant with respect to a group of rotations was constructed independently.[2] A solution of this type for the Navier-Stokes equations was found to be reducible to the classical invariant solution defining the radial fluid flow from a source.[3] A singular vortex has been investigated in detail[4–7] with the object of its physical content when there are additional constraints: of homogeneity, stationarity, etc. An analogous solution with a physical interpretation of the flow being described was constructed[8–10] for a model of ideal magnetohydrodynamics. A singular vortex has been described[11] for the equations of ideal Tresca plasticity.

The general concept of a singular vortex was introduced by Ovsyannikov in May 2004. In particular, it was noted that, in spite of the name which has been adopted for it, a singular vortex can also be irrotational, and this means that a vector field which satisfies the condition of partial invariance with respect to a group of rotations can have zero curl. Finite implicit formulae, defining an irrotational singular vortex, have been given, but no geometrical treatment of the implicit functional relation has not been presented.

This paper aims to fill this gap. Furthermore, it will be shown that an irrotational field of the singular vortex type cannot be realized as the velocity field for the motion of a continuum which satisfies the law of conservation of mass.

1. Implicit relation defining a singular vortex

A spherical system of coordinates (r,θ,φ) (the value $\theta=0$ corresponds to the north pole) is used. The vector u is represented in the form of the sum of a radial component U and a component tangential to the spheres $r=\text{const}$. This last component is described either using its projections V and W on to directions which are tangential to the meridians and parallels of a sphere or using the modulus of the tangential component of H and the angle of inclination from the meridian ω:

$$V = H\cos\omega, \quad W = H\sin\omega \tag{1.1}$$

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The group of rotations $SO(3)$ is generated in the space $\mathbb{R}^3(\mathrm{x}) \times \mathbb{R}^3(\mathrm{u})$ by the infinitesimal operators:

$$X_7 = -\sin\varphi\partial_\theta - \cos\varphi\,\mathrm{ctg}\,\theta\partial_\varphi + \frac{\cos\varphi}{\sin\theta}\partial_\omega$$

$$X_8 = \cos\varphi\partial_\theta - \sin\varphi\,\mathrm{ctg}\,\theta\partial_\varphi + \frac{\sin\varphi}{\sin\theta}\partial_\omega, \quad X_9 = \partial_\varphi$$

The invariants of the group $SO(3)$ are the coordinate r and the functions U and H. The required vector field of the singular vortex type, which is partially invariant with respect to the $SO(3)$ group, is distinguished by the relations

$$U = U(r), \quad V^2 + W^2 = H^2(r), \quad \omega = \omega(\mathbf{x}) \tag{1.2}$$

Hence, the invariant functions U and H are assume to depend solely on the invariant independent variable r while the non-invariant function ω depends on all the initial independent variables (r,θ,φ). It is further assumed that $H \neq 0$, that is, the thoroughly studied centrally symmetric vector field $V = W = 0$ is excluded from the treatment.

We shall investigate the vector fields u in a three-dimensional space, defined by Eq. (1.2) and the condition $\mathrm{rot}\,\mathrm{u} = 0$. These two conditions are equivalent to the following system of equations

$$(rV)_r = 0, \quad (rW)_r = 0, \quad V_\varphi = (\sin\theta W)_\theta \tag{1.3}$$

The representations

$$H = \alpha/r, \quad \omega = \omega(\theta, \varphi)$$

with an arbitrary constant $\alpha \neq 0$ follow from this and from relations (1.1). The equation for the function ω is obtained from the last relation of (1.3):

$$\sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi = -\cos\theta\sin\omega \tag{1.4}$$

The general integral of this equation in the form

$$\eta = f(\zeta); \text{ where } \eta = \sin\theta\sin\omega, \quad \zeta = \varphi + \pi/2 + \mathrm{arctg}(\cos\theta\,\mathrm{tg}\,\omega) \tag{1.5}$$

determines the implicit dependence $\omega = \omega(\theta,\varphi)$.

The following assertion therefore holds.

Assertion 1 (L. V. Ovsyannikov). An irrotational vector field that is partially invariant with respect to the $SO(3)$ group of rotations is represented in a spherical system of coordinates in the form

$$U = U(r), \quad V = \alpha r^{-1}\cos\omega(\theta, \varphi), \quad W = \alpha r^{-1}\sin\omega(\theta, \varphi) \tag{1.6}$$

Here $U(r)$ and α are an arbitrary function and a constant. The quantity ω is implicitly defined by Eq. (1.5) with an arbitrary smooth function f.

Remark. The invariants η and ζ were chosen by Ovsyannikov in a somewhat different form. However, the representation given here is more convenient for the geometrical treatment obtained below.

There are two arbitrary functions in solution (1.5), (1.6). The choice of the function U defines the radial component of the vector field, and its effect on the resulting vector field is obvious. The function f characterizes the component of the vector field which is tangential to the spheres $r = \mathrm{const}$, and its geometric treatment is explained below.

2. Geometrical treatment of the implicit relation

Implicit relation (1.5) defines the function ω for the deviation of the tangential component of a vector field u from the meridian, and this means that it defines a certain tangential field of directions on a sphere. We shall assume that it is already known. Suppose a point M, is defined by the angles θ and φ of the spherical system of coordinates. Besides the main system of coordinates $Oxyz$, an auxiliary system $O\xi_1\xi_2\xi_3$ is introduced into the treatment according to the following rule: the $O\xi_3$ axis is chosen to pass through the point M, the $O\xi_1$, axis is collinear with the tangential vector field being considered at the point M and the $O\xi_2$ axis is defined such that the resulting system of orthogonal coordinates is a right-handed system (see Fig. 1). For an arbitrary point on the sphere, its radius vector in the system of coordinates $Oxyz$ is denoted by r and, in the system of coordinates $O\xi_1\xi_2\xi_3$, by R.

The relation between the triplets of numbers r and R is given by the conversion rule

$$\mathbf{R} = \Gamma_{12}(\omega)\Gamma_{13}(\theta)\Gamma_{12}(\varphi)\mathbf{r}, \quad |\mathbf{r}| = |\mathbf{R}| = 1 \tag{2.1}$$

Here, φ, θ and ω are the Euler angles, and Γ_{12} and Γ_{13} are the orthogonal operators

$$\Gamma_{12}(\alpha) = \begin{Vmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \qquad \Gamma_{13}(\alpha) = \begin{Vmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{Vmatrix} \tag{2.2}$$

The vector $R_* = (0,1,0)$ is considered. According to formulae (2.1), in the initial system of coordinates it has the form

$$\mathbf{r}_* = (a, b, \eta) \tag{2.3}$$

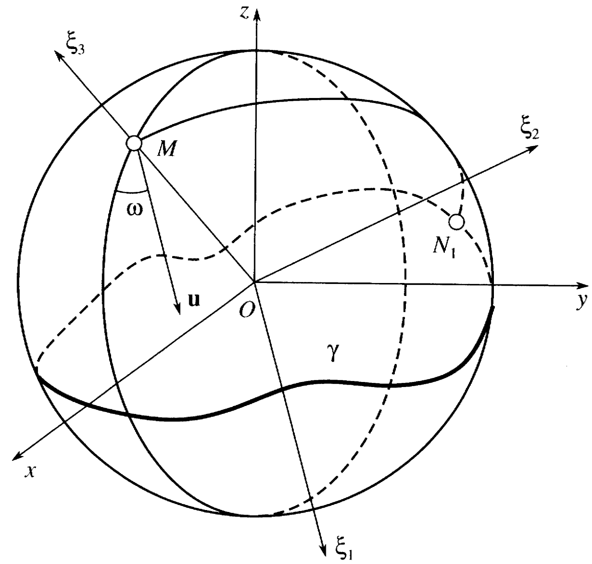

Fig. 1.

where

$$a = -\cos\omega\sin\varphi - \sin\omega\cos\theta\cos\varphi, \quad b = \cos\omega\cos\varphi - \sin\omega\cos\theta\sin\varphi$$

and the component η is defined by the second formula of (1.5). At the same time, according to the last formula of (1.5)

$$\cos\zeta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\zeta = \frac{b}{\sqrt{a^2+b^2}}$$

It follows from this that the point defined by the radius vector r_* lies on the unit sphere, it has been raised above the Oxy plane by an amount η and the projection of the vector r_* onto the Oxy-plane makes an angle ζ with the positive direction of the Ox axis. The relation between the quantities η and ζ is given by Eq. (1.5).

3. An algorithm for constructing an irrotational singular vortex

The following treatment of relation (1.5), which is based on the results of the preceding section, holds. Suppose a curve γ on the sphere $r=1$ is define by relation (1.5) according to the following rule: η is the height of a point above the Oxy plane and ζ is the angle between the projection of the radius vector of the point onto the Oxy plane and the positive direction of the Ox axis.

The algorithm for finding the function ω at the point M is as follows (see Fig. 1).

1. A circumference S^1, each point of which is at a geodesic distance $\pi/2$ from the point M, is drawn for the given point M on the sphere S^2. Suppose $N_1,\cdots,N_k$ are the points of intersection of the curve γ with the circumference S^1 constructed.
2. For each geodesic going from the point M to a point N_i, its tangential vector at the point M is rotated clockwise through an angle of $\pi/2$ in the tangential plane.
3. The angles between the resulting vectors and the meridian passing through the point M define the required values of the angle ω.

Assertion 2. The above algorithm gives all possible values of the angle ω which satisfy relation (1.5).

The proof of this assertion follows from the fact that the geometrical interpretation, presented in Section 2, holds for each value of ω satisfying relation (1.5). Consequently, this value can be obtained using the above mentioned algorithm.

4. The geometrical properties of the resulting direction fields

Suppose a curve γ on the sphere $|r|=1$ is specified parametrically in the form $r=r(s)$ with a natural parameter s. The vector $\dot{r}$ is tangential to γ, and this means that it is also tangential to S^2. Together with the vector $b = r \times \dot{r}$, they form an orthonormal basis in three-dimensional space.

The field of directions on the sphere S^2, which is determined by the algorithm presented in Section 3, can be constructed in the following way. A certain value $s=s_*$ is stipulated and the point $r(s_* \in \gamma)$ corresponds to it. A geodesic of length π, connecting the points $-b(s_*)$ and $b(s_*)$ and passing through the point $\dot{r}(s_*) \in S^2$, is drawn on the sphere. For each point of this geodesic the proposed algorithm gives the preselected point $r(s_*)$ as one of the possible points of intersection N_i (the notation of Section 3 is used). The required direction field is tangential to the geodesic. The construction carried out above is repeated for all possible values of the parameter s. This algorithm enables one to construct the direction field on the sphere over the whole of its domain of definition.

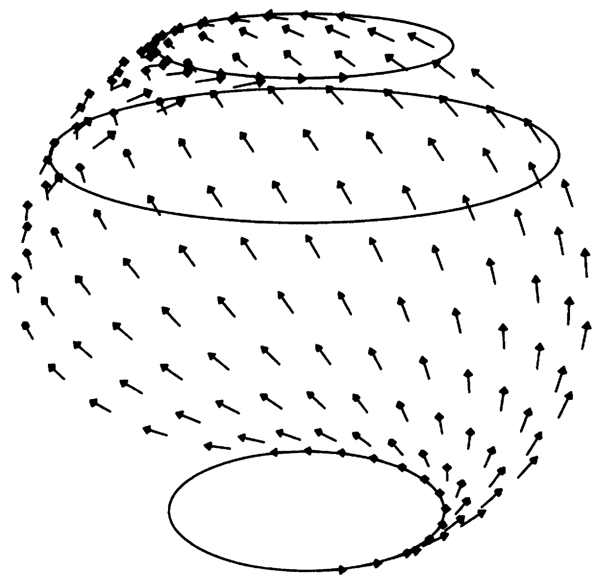

Fig. 2.

The equidistant curves $\gamma^{\pm}(\pi/2)$ of the curve γ, that is, the sets of points which have been shifted along the normal geodesic by a distance $\pm\pi/2$ with respect to γ, serve as the boundaries of the domain of definition of the direction field. The equidistant curves $\gamma^{\pm}$ are parametrically defined by the formula

$$\mathbf{r} = \pm\mathbf{b}(s)$$

At the points of the equidistant curves $\gamma^{\pm}$, the direction field is arranged along a tangent to them. The cuspidal points on the equidistant curves correspond to the values of the parameter s for which the acceleration vector $\ddot{r}$ is collinear with the vector r. The geodesic curvature of the curve γ vanishes at these points. The direction field is not uniquely defined at the cuspidal points and, moreover, it is impossible to choose a branch of the function ω which is continuous on passing through a cuspidal point of an equidistant curve. Analogous singularities arise in domains on the sphere S^2 which are bounded by self-intersecting segments of the equidistant curves $\gamma^{\pm}$. Along any path passing through a domain on the sphere bounded by a self-intersecting part of the equidistant curve, the function ω has a discontinuity on one of the boundaries of this domain.[10]

The field directed along the meridians of the sphere is the unique smooth direction field, determined by the algorithm presented in Section 3, on the whole sphere without the poles which is without singularities. The equator of the sphere serves as the curve γ for it.

The direction field, shown in Fig. 2 for $\theta_* = \pi/2$, is obtained when the parallels $\theta = \theta_*, 0 \le \theta_* \le \pi/2$ are chosen as the curve γ. For convenience of perception, the direction field on just a hemisphere is presented. Its domain of definition is the strip between the equidistant curves $\gamma^- : \theta = \pi/2 - \theta_*$ and $\gamma^- : \theta = \pi/2 + \theta_*$. The vector field is undefined for points from the neighbourhoods of the poles, that is, when $\theta < \pi/2 - \theta_*$ and $\theta > \pi/2 + \theta_*$, since the geodesic distance from these points to the parallel $\theta = \theta_*$ is greater than $\pi/2$. At the points of the equidistant curves $\gamma^{\pm}$, the field is directed along the tangent to the equidistant curves.

Another example of a direction field is shown in Fig. 3, where, for convenience, the direction field is also only shown on a heini sphere. The "perturbed equator" of the sphere, that is, the curve $\theta = \pi/2 + (\sin 5\varphi)/8$, was chosen as the curve γ. The geodesic curvature of γ vanishes when $\varphi = k\pi/5 (k = 0,1,\cdots,9)$. Hence, the equidistant curves have ten cuspidal points. By virtue of the periodicity of the curve $\gamma^{\pm}$, the equidistant curves $\gamma^{\pm}$ are drawn twice for a variation in $\varphi \in [0,2\pi]$, and the cuspidal points are therefore pairwise identical. The equidistant curves have a star-like shape (Fig. 3). Within the domains bounded by the equidistant curves in the polar regions, the direction field is defined non-uniquely and, furthermore, it is impossible to choose a branch of the function ω which continuously passes through all the boundaries of these domains.

5. Compatibility with the equation of continuity

In an Ovsyannikov vortex in the case of the equations of gas dynamics,[1] the velocity field of the gas has the representation (1.2) (a time-dependence of these functions is also permitted) and the density and pressure depend solely on the time t and the radial coordinate r. The representation is dictated by the requirement of the partial invariance of the solution with respect to the $SO(3)$ group of rotation.

A similar representation of the solution holds in magnetohydrodynamics[8] and in the dynamics of a viscous fluid.[3]

It is proved below that the irrotational vector fields of the singular vortex type, which have been constructed cannot be realized in the form of the velocity fields of flows of a continuum which obeys the equation of continuity with a density depending solely on t and r. In fact, if the velocity field of a continuum is defined by a vector field described by Assertion 1 and the density is invariant with respect to a group of rotations, that is, $\rho = \rho(t,r)$, then the equation of continuity has the form

$$\rho^{-1}(\rho_t + U\rho_r) + r^{-2}(r^2 U)_r + (r\sin\theta)^{-1}(\cos\theta V + \sin\theta V_\theta + W_\varphi) = 0$$

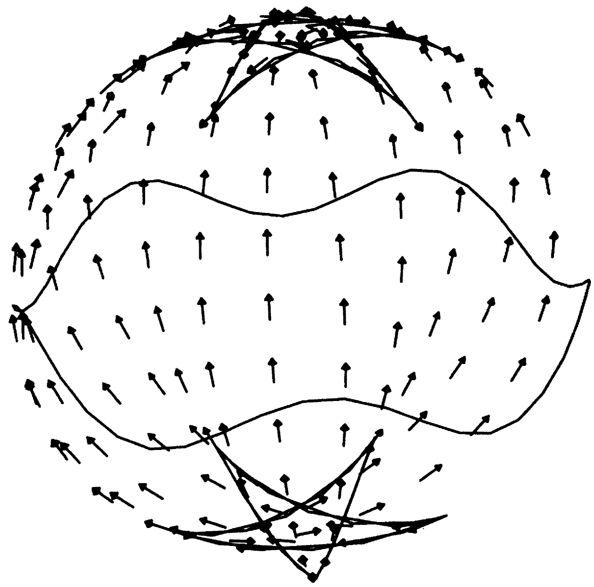

Fig. 3.

By substituting expressions (1.6) for the components V and W we can separate the variables in this equation. Introducing the new invariant function

$$h(t,r) = \alpha^{-1}r^2(\rho^{-1}(\rho_t + U\rho_r) + r^{-2}(r^2U)_r)$$

we have

$$\sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi = \cos\theta\cos\omega + h\sin\theta \tag{5.1}$$

It remains to verify that Eq. (5.1) is compatible with Eq. (1.4). Expressing the derivatives of the function ω from these two equations, we obtain

$$\omega_\theta = h\sin\omega, \quad \omega_\varphi = -h\sin\theta\cos\omega - \cos\theta$$

Cross differentiation and substitution of the derivatives give the contradictory equality

$$(1+h^2)\sin\theta = 0$$

The following assertion therefore holds.

Assertion 3. The motion of a continuum of the singular vortex type is substantially rotational, that is, the curl of the velocity is non-zero in it.

6. Conclusion

Assertion 1, in conjunction with the geometrical treatment of relation (1.5), gives an exhaustive representation of the structure of three-dimensional irrotational vector fields which are partially invariant with respect to the $SO(3)$ group of rotations. In a spherical system of coordinates, the radial component of such fields depends arbitrarily on the distance r to the origin of coordinates. The component tangential to the spheres r = const is proportional with respect to its modulus to r^{-1} and is determined with respect to its direction by a certain direction field on a sphere. This direction field is constructed with a functional arbitrariness using the algorithms given in Sections 3 and 4. The general arbitrariness in the solution is two functions of a single argument and a single constant.

We have shown that an irrotational vector field, which is partially invariant with respect to the $SO(3)$ group of rotations, cannot be realized as the velocity field of a continuum which obeys the equation of continuity in the case of a density that depends solely on time and the radial coordinate. Hence, the term “singular vortex” that is used for exact $SO(3)$ solutions is completely justified for partially invariant solutions of the equations of continuum mechanics.

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